Bollettino Unione Matematica Italiana

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On a generalized Hermite polynomial.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **12** (1957), n.2, p. 241–246.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1957_3_12_2_241_0>

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1957.

On a generalized Hermite polynomial.

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Sunto. - *We obtain several relations concerning generalized* HERMITE *polynomials which have studied recently by* TOSCANO.

1. TOSCANO [8] studied the polynomial defined by

$$G_{n,\nu}(x) = xG_{n-1,\nu}(x) - (n+\nu)G_{n-2,\nu}(x) \qquad (n \geq 2) \tag{1.1}$$

$$G_{0,\nu}(x) = 1, \quad G_{1,\nu}(x) = x.$$

Obviously for $\nu = -1$ we have the HERMITE polynomial, and for $\nu = 0$ we have polynomial associated with the HERMITE polynomial.

The purpose of this note is to obtain further results concerning these polynomials. In particular we shall obtain an analog of a formula of Nielsen (see formula 2.1 below). We shall also show that the derivate of $G_{n,\nu}(x)$ satisfy the TURAN inequality.

2. First we prove the formula

$$H_m(x)G_{n,\nu}(x) = \sum_{r=0}^{m} r!\binom{m}{r}\binom{n+\nu+1}{r} G_{n+m-2r,\nu}(x) \tag{2.1}$$

where $H_m(x)$ is the HERMITE polynomial and $m \leq n$.

The special cases of (2.1) for $\nu = -1$ and $v = 0$ were given by NIELSEN [3]. The case $\nu = -1$ was rediscovered by DHAR [3] and FELDHEIM [4]. Recently CARLITZ [1] proved by induction that if $U_\mu(x)$ is any solution of

$$U_{\mu+1}(x) = xU_\mu(x) - \mu U_{\mu-1}(x)$$

then

$$H_m(x)U_\mu(x) = \sum_{r=0}^{m} r!\binom{m}{r}\binom{\mu}{r} U_{m+\mu-2r}(x)$$

where μ is an arbitrary complex number.

To prove (2.1) we shall employ the method of CARLITZ. Indeed

(2.1) is true for $m=0$. Assume it is true for $m=k$, and consider

$$H_{k+1}(x)G_{n,\nu}(x)=\{xH_k(x)-kH_{k-1}(x)\}\,G_{n,\nu}(x)=$$

$$=\sum_{r=0}^{k} r!\binom{k}{r}\binom{n+\nu+1}{r}xG_{n+k-2r,\nu}(x)-$$

$$-\sum_{r=0}^{k} r!\binom{k}{r}\binom{n+\nu+1}{r}(k-r)G_{n+k-1-2r,\nu}(x)=$$

$$=\sum_{r=0}^{k} r!\binom{k}{r}\binom{n+\nu+1}{r}\{xG_{n+k-2r,\nu}(x)-(n+k+\nu+1-2r)\cdot G_{n+k-1-2r,\nu}(x)\}+$$

$$+\sum_{r=0}^{k} r!\binom{k}{r}\binom{n+\nu+1}{r}(n+\nu+1-2r)G_{n+k-1-2r,\nu}(x)=$$

$$=\sum_{r=0}^{k} r!\binom{k}{r}\binom{n+\nu+1}{r}G_{n+k+1-2r,\nu}(x)+$$

$$+\sum_{r=1}^{k+1} r!\binom{k}{r-1}\binom{n+\nu+1}{r}G_{n+k+1-2r,\nu}(x)=$$

$$=\sum_{r=0}^{k+1} r!\binom{k+1}{r}\binom{n+\nu+1}{r}G_{n+k+1-2r,\nu}(x)$$

which completes the proof.

In a similar fashion we can prove that the inverse formula of (2.1) is

$$G_{m+n,\nu}(x)=\sum_{r=0}^{m}(-1)^r r!\binom{m}{r}\binom{n+\nu+1}{r}H_{m-r}(x)G_{n-r,\nu}(x) \tag{2.2}$$

where $m\leq n$.

We point out here that (2.1) and (2.2) can be proved without the assumption that n is an integer. Indeed we can replace $G_{n,\nu}(x)$ by any solution $U_{\mu,\nu}(x)$ of the difference equation (1.1).

Similarly the formula

$$\frac{H_m(x)G_{n+1,\nu}(x)-H_{m+1}(x)G_{n,\nu}(x)}{(m-n-\nu-1)}= \tag{2.3}$$

$$=\sum_{r=0}^{m} r!\binom{m}{r}\binom{n+\nu+1}{r}G_{n+m-1-2r,\nu}(x)$$

can be proved by induction. This formula reduces to a formula of CARLITZ's for $\nu=-1$.

3. Now we shall prove the formula

$$G'_{n+1,\nu}(x)G_{n,\nu}(x) - G_{n+1,\nu}(x)G'_{n,\nu}(x) = \tag{3.1}$$
$$= \sum_{r=0}^{n} r!\binom{n+\nu+1}{r} G^{*}_{n-r,\nu}(x)$$

where $G'_{n,\nu}(x) = \frac{d}{dx} G_{n,\nu}(x)$.

From (1.1) we have

$$G_{n+1,\nu}(x) = xG_{n,\nu}(x) - (n+\nu+1)G_{n-1,\nu}(x)$$
$$G_{n+1,\nu}(y) = yG_{n,\nu}(y) - (n+\nu+1)G_{n-1,\nu}(y).$$

Hence

$$\frac{G_{n+1,\nu}(x)G_{n,\nu}(y) - G_{n,\nu}(x)G_{n+1,\nu}(y)}{x-y} = \tag{3.2}$$

$$= G_{n,\nu}(x)G_{n,\nu}(y) + (n+\nu+1)\frac{G_{n,\nu}(x)G_{n-1,\nu}(x) - G_{n-1,\nu}(x)G_{n,\nu}(y)}{x-y} =$$
$$= \sum_{r=0}^{n} r!\binom{n+\nu+1}{r} G_{n-r,\nu}(x)G_{n-r,\nu}(y).$$

Now (3.1) follows from (3.2) if we let $x - y$.

4. For a sequence of functions $\{f_n(x)\}$, let

$$\Delta_n(x) = [f_n(x)]^2 - f_{n-1}(x)f_{n+1}(x).$$

Szegö [7] proved that $\Delta_n(x) \geq 0$ for the Legendre, Hermite, and ultraspherical polinomials in certain regions. Several other authors extended this result to other functions.

New define

$$D_n(x) = [G'_{n,\nu}(x)]^2 - G'_{n+1,\nu}(x)G'_{n-1,\nu}(x),$$
$$D_0(x) = 0.$$

If we differentiate formula (1.1) we get

$$G'_{n,\nu}(x) = xG'_{n-1,\nu}(x) + G_{n-1,\nu}(x) - (n+\nu)G'_{n-2,\nu}(x)$$

and, if we replace n by $n+1$,

$$G'_{n+1,\nu}(x) = xG'_{n,\nu}(x) + G_{n,\nu}(x) - (n+\nu+1)G'_{n-1,\nu}(x)$$

Multiplying the first by $G'_{n,\nu}(x)$ and the second by $G'_{n-1,\nu}(x)$ and then subtracting the resulting equations we get

$$D_{n+1}(x) = [G'_{n+1,\nu}(x)G_{n,\nu}(x) - G'_{n,\nu}(x)G_{n+1,\nu}(x)] + \\ + (n+\nu+1)D_n(x) + [G'_{n,\nu}(x)]^2.$$

Now by (3.1)

$$D_{n+1}(x) = \sum_{r=0}^{n} r!\binom{n+\nu+1}{r}G^2_{n-r,\nu}(x) + [G'_{n,\nu}(x)]^2 + (n+\nu+1)D_n(x).$$

From this we see that

$$D_n(x) \geq 0 \qquad (n \geq 1,\ -\infty < x < +\infty). \tag{4.1}$$

In fact we have the explicit formula

$$D_{n+1}(x) = \sum_{r=0}^{n} r!\binom{n+\nu+1}{r}[G'_{n-r,\nu}(x)]^2 + \\ + \sum_{s=0}^{n} s!\binom{n+\nu+1}{s}\sum_{r=0}^{n-s} r!\binom{n-s+\nu+1}{r} G^2_{n-r-s,\nu}(x).$$

Interchanging the order of summation in the double sum, we finally get

$$D_{n+1}(x) = \sum_{r=0}^{n}\binom{n+\nu+1}{r}\{[G'_{n-r,\nu}(x)]^2 + (r+1)G^2_{n-r,\nu}(x)\} \tag{4.2}$$

Toscano proved

$$\Delta_{n+1}(x) = (1+\nu)_{n+1} + \sum_{r=0}^{n} r!\binom{n+\nu+1}{r} G^2_{n-r,\nu}(x).$$

Hence

$$D_{n+1}(x) = \Delta_{n+2}(x) - (1+\nu)_{n+1} \\ + \sum_{r=0}^{n} r!\binom{n+\nu+1}{r}\{[G'_{n-r,\nu}(x)]^2 + rG^2_{n-r,\nu}(x)\}. \tag{4.3}$$

But the last term in the series on the right hand side of (4.3) is greater than $(1+\nu)_{n+1}$. Hence we obtain

$$D_n(x) > \Delta_n(x) \qquad (n \geq 1,\ -\infty < x < +\infty). \tag{4.4}$$

We also note here that (4.3) reduces for the case $\nu = -1$ to the relation

$$D_{n+1}(x) = \Delta_{n+1}(x) + n(n+1)\Delta_n(x)$$

which was obtained previously [1] by the present author for the ordinary Hermite polynomials.

5. Let us now consider the expression

$$\delta_n(x) = [G_{n,\nu}(x)]^2 - G''_{n-1,\nu}(x)G''_{n-1,\nu}(x) \qquad (n \geq 2),$$
$$\delta_0(x) = \delta_1(x) = 0,$$

where $G''_n(x) = \frac{d^2}{dx^2} G_n(x)$.

In this section we shall prove

$$\delta_n(x) > 0 \qquad (n \geq 2, \ -\infty < x < +\infty), \tag{5.1}$$

First we require the following lemmas:

Lemma 1.

$$\Gamma_n(x) = [G'_{n,\nu}(x)]^2 - G_{n,\nu}(x)G''_{n,\nu}(x) > 0 \quad (\nu > -2, \ -\infty < x < +\infty)$$

This follows from the fact that $G_{n,\nu}(x)$ are orthogonal polynomials for $\nu > -2$ and hence has all its zeres real. But [5] any polynomial $p_n(x)$ with real zeroes satisfy

$$[p'_n(x)]^2 - p_n(x)p''_n(x) > 0.$$

Hence

$$\Gamma_n(x) > 0.$$

Lemma 2. - If

$$\lambda_{n+1}(x) = G''_{n+1,\nu}(x)G'_{n,\nu}(x) - G'_{n+1,\nu}(x)G''_{n,\nu}(x)$$

then

$$\lambda_{n+1}(x) = \sum_{k=0}^{n} k!\binom{n+\nu+1}{k} \{ [G'_{n-k,\nu}(x)]^2 + \Gamma_{n-k}(x) \} \tag{5.2}$$

and hence

$$\lambda_{n+1}(x) > 0 \qquad (n \geq 2, \ -\infty < x < +\infty).$$

Proof. - If we differentiate the recurrence relation (1.1) once then we get easily

$$\frac{G'_{n+1,\nu}(x)G'_{n,\nu}(y) - G'_{n,\nu}(x)G'_{n+1,\nu}(y)}{x-y} = G'_{n,\nu}(x)G'_{n,\nu}(x) +$$
$$+ \frac{G'_{n,\nu}(y)G_{n,\nu}(x) - G_{n,\nu}(y)G'_{n,\nu}(x)}{x-y} +$$
$$+ (n+\nu+1)\frac{G'_{n,\nu}(x)G'_{n-1,\nu}(y) - G'_{n,\nu}(x)G'_{n-1,\nu}(x)}{x-y} =$$
$$= \sum_{k=0}^{n} k!\binom{n+\nu+1}{k} \Big\{ G'_{n-k,\nu}(x)G'_{n-k,\nu}(y) +$$
$$+ \frac{G_{n-k,\nu}(x)G'_{n-k,\nu}(y) - G_{n-k,\nu}(y)G'_{n-k,\nu}(x)}{x-y} \Big\}.$$

Taking the limit as $x \to y$, formula (5.2) follows.

Now to prove (5.1), differentate (1.1) twice. We obtain

$$\delta_{n+1}(x) = (n + \nu + 1)\delta_n(x) + [G''_{n,\nu}(x)]^2 + 2\lambda_{n+1}(x) =$$

$$= (n + \nu + 1)\delta_n(x) + [G''_{n,\nu}(x)]^2 + 2 \sum_{k=0}^{n} k! \binom{n+\nu+1}{k} \cdot$$

$$\cdot \{ [G'_{n-k,\nu}(x)]^2 + \Gamma_{n-k}(x) \}.$$

Hence

$$\delta_{n+1}(x) = \sum_{k=0}^{n} k! \binom{n+\nu+1}{k} \{ (2k+2)[G'_{n-k,\nu}(x)]' + \tag{5.3}$$

$$+ (2k+2)\Gamma_{n-k}(x) + [G''_{n-k,\nu}(x)]^2 \}$$

and formula (5.1) follows from (5.3).

REFERENCES

[1] W. A. AL-SALAM, *On characterization of some orthogonal functions,* « American Mathematical Monthly », vol. 64, 1957, pp. 29 32.

[2] L, CARLITZ, *A note on Hermite polynomials,* « American Mathematical Monthly », vol. 62, 1955, pp. 646-647.

[3] S. DHAR, *On the product of parabolic cylinder functions,* « Bulletin of the Calcutta Math. Soc. », vol. 26, 1934, pp. 57-64.

[4] E, FELDHEIM, *Quelques nouvelles relations pour les polinomes d'Hermite,* « Journal of the London Math. Soc. », vol. 13, 1938, pp. 22-29.

[5] B. N. MUKHERJEE and T. S NANJUNDIAH, *On an inequality relating to Laguerre and Hermite polynomials,* « The Mathematics Student », vol. XIX, 1951, pp. 47-48.

[6] N. NIELSEN, *Recherches sur les polinomes d'Hermite,* « Det. hgl. Danske Videnshabernes Selskab, Mathematisk-fysiske Meddelelser », 1, vol. 6, 1918, pp. 1-78.

[7] G. SZEGÖ, *On an inequality of P. Turan concerning the Legendre po-polynomials,* « Bulletin of the American Mathematical Soc. », vol. 54, 1948. pp. 401-405.

[8] L. TOSCANO, *Formule di riduzione tra funzioni e polinomi classici.* « Rivista Univ. Parma », vol. 6, 1955, pp. 117-140.