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On the Bessel polynomials.

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On the Bessel Polynomials.

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Sunto. - *We derive a formula for the product of two consecutive* BESSEL *polynomials. We use this relation to characterize those polynomials.*

KRALL and FRINK [1] defined the BESSEL polynomials $y_n(x)$ as the solution of the differential equation

$$(1) \qquad x^2 \frac{dy^2(x)}{dx^2} + (2x+2)\frac{dy(x)}{dy} = n(n+1)y(x) \qquad (n, \text{ integer})$$

which has the value 1 at $x = 0$. They gave among other results

$$(2) \qquad y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!\,k!}\left(\frac{x}{2}\right)^k$$

$$(3) \qquad x^2 y_n'(x) = (nx-1)y_n(x) + y_{n-1}(x)$$

$$(4) \qquad x^2 y'_{n-1}(x) = y_n(x) - (nx+1)y_{n-1}(x).$$

They remark that the second solution of (1) is $e^{2/x}y_n(-x)$.

In this note we give recurrence relations for the second solution corresponding to (3) and (4), we shall derive a formula for the product of two consecutive polinomials and a corresponding one for the second solution, and finally give a characterization of these polynomials based on this formula.

If we multiply (3) by $y_{n-1}(x)$ and (4) by $y_n(x)$ and then add the two equations we get our first result

$$(5) \qquad x^2 \frac{d}{dx}\left(y_n(x)y_{n-1}(x)\right) = \left(y_n(x) - y_{n-1}(x)\right)^2.$$

Consider now the function $q_n(x) = (-1)^n e^{2/x} y_n(-x)$ which is a

(1) H. L. KRALL and O. FRINK, *A new class of orthogonal polynomials, the Bessel polynomials.* Trans. Am. Math. Soc. Vol. 65, (1949), pp. 100-115.

second solution of (1). Differentianting $q_n(x)$ and applying (3) we find

$$x^2 q_n'(x) = (nx - 1)q_n(x) + q_{n-1}(x). \tag{3'}$$

Similarly

$$x^2 q'_{n-1}(x) = q_n(x) - (nx + 1)q_{n-1}(x) \tag{4'}$$

We find from (3') and (4') a similar relation to (5)

$$x^2 \frac{d}{dx}[q_n(x)q_{n-1}(x)] = [q_n(x) - q_{n-1}(x)]^2 \tag{5'}$$

We now give a characterization of the BESSEL polynomials. We prove the following theorem.

THEOREM. - *Given a sequence of polynomials* $\{f_n(x)\}$ *where* $deg\, f_n = n$ *and* $f_0(x) = 1$, *then* $f_n(x) = y_n(x)$ *if and only if*

$$x^2 \frac{d}{dx}[f_n f_{n-1}] = [f_n - f_{n-1}]^2. \tag{6}$$

PROOF. - We first remark that (6) and the condition $f_0 = 1$ imply that the constant term in every f_n is 1. We also remark that the coefficient of highest power of x in the polynomial $f_n(x)$ satisfying (6) must necessarily be $\frac{(2n)!}{n!\,2^n}$.

Let $f_1(x) = ax + 1$ and substitute in (6) getting $x^2 a = a^2 x^2$. Hence $f_1(x) = x + 1 = y(x)$. Therefore the theorem is true for $n = 1$.

Now assume the theorem true for all n up to $k-1$. We assert that it is true for $n = k$. For otherwise let $f_k(x) = y_k(x) + w(x)$ where $w(x) = c_1 x^r + c_2 x^{r-1} + \ldots$, $r \leq k$. From (6)

$$x^2 \frac{d}{dx}[f_k f_{k-1}] = x^2 \frac{d}{dx}[f_{k-1} y_k] + x^2 \frac{d}{dx}[f_{k-1} w] = [f_{k-1} - y_k - w]^2$$

which according to the inductive hypothesis becomes

$$x^2 \frac{d}{dx}[y_{k-1} w] = w^2 - 2w(y_{k-1} - y_k).$$

Equating coefficients of highest power of x on both sides of the equation, we get in case $r = k$, $c_1 = 0$. In case $r < k$, we get

$$\frac{(2k-2)!\,(k+r-1)}{2^{k-1}(k-1)!} c_1 = \frac{(2k)!\,c_1}{k!\,2^k} \quad \text{or} \quad r = 3k - 1$$

which is a contradiction. Hence the theorem is established.

COROLLARY. - Given a sequence of functions $\{f_n(x)\}$ where $f_n(x) = e^{2/x}v_n(x)$ and $v_n(x)$ is a polynomial of degree n and $v_0(x) = 1$ and such that

$$x^2 \frac{d}{dx}(f_n f_{n-1}) = (f_n - f_{n-1})^2 \tag{7}$$

then $v_n(x) = (-1)^n y_n(-x)$.

PROOF. - From (7) we find $x^2 \dfrac{d}{dx}(v_n v_{n-1}) = (v_n(x) + v_{n-1}(x))^2$. Multiplying through by $(-1)^{2n-1}$ and change x to $-x$ we find

$$x^2 \frac{d}{dx}[(-1)^n v_n(-x) \cdot (-1)^{n-1} v_{n-1}(-x)] =$$
$$= [(-1)^n v_n(-x) - (-1)^{n-1} v_{n-1})]^2.$$

And hence by the previous theorem since $v_0 = 1$ we have

$$(-1)^n v_n(-x) = y_n(x) \quad \text{or} \quad v_n(x) = (-)^n y_n(-x).$$

COROLLARY. - Given a sequence of functions $\{f_n(x)\}$ where $f_n(x) = (-1)^n g(x) y_n(-x)$ and satisfying

$$x^2 \frac{d}{dx}(f_n f_{n-1}) = (f_n - f_{n-1})^2 \tag{8}$$

then $g(x) = Ce^{2/x}$.

PROOF. - Substituting for f_n in (8) we get after reduction

$$-x^2 g^2(x) \frac{d}{dx}[y_n(-x) y_{n-1}(-x)] - 2x^2 g(x) g'(x) y_n(-x) y_{n-1}(-x) =$$
$$= g^2(x)[y_n^2(-x) + y^2_{n-1}(-x) + 2y_n(-x) y_{n-1}(-x)].$$

But from the theorem above

$$-x^2 \frac{d}{dx}[y_n(-x) y_{n-1}(-x)] = y_n^2(-x) + y^2_{n-1}(-x) - 2y_n(-x) y_{n-1}(-x).$$

Hence $-2x^2 g(x) g'(x) = 4g^2(x)$, whose solution is $g(x) = Ce^{2/x}$.