Bollettino Unione Matematica Italiana

Aurel Wintner

On stable oscillations of high frequency.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **12** (1957), n.1, p. 9–11.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1957_3_12_1_9_0>

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1957.

On stable oscillations of high frequency.

Nota di AUREL WINTNER (Baltimore, U. S. A.)

Sunto. - *It is well-known that if a continuous function,* ω(t), *defined for large positive* t, *tends to* ∞ *as* t → ∞, *then all solutions of* $d^2x/dt^2 + \omega^2(t)x = 0$ *stay bounded as* t → ∞, *provided that* ω(t) *is monotone. This is not in general true if the latter proviso is omitted. The purpose of this note is the specification of such perturbations of the monotone behavior of* ω(t) *as are «small» enough to preserve the boundedness of all solutions when* $\omega(\infty) = \infty$.

Let $\omega = \omega(t)$ be a function which is continuous for $0 \leqq t < \infty$, and let the corresponding differential equation

$$d^2x/dt^2 + \omega^2(t)x = 0 \tag{1}$$

be called stable if every solution $x = x(t)$ is $O(1)$ as $t \to \infty$. It is well-known (cf., e. g., [2], pp. 28-29) that if $\omega(t)$ is non-decreasing and has a positive lower bound, then, with reference to any real-valued, non-trivial ($\not\equiv 0$) solution $x(t)$ of (1), the local maxima of $|x(t)|$ form a non-increasing sequence. In particular, (1) will be stable if $d\omega(t) \geqq 0$ and

$$\omega(t) \to \infty \quad \text{as} \quad t \to \infty. \tag{2}$$

In certain applications, it was often considered evident that (2) *alone* (that is, *without* $d\omega(t) \geqq 0$) will suffice. But it was shown in [4] that this stability criterion is false. Thus there arises the need for the specification of such perturbations of the monotone behavior of $\omega(t)$ as are «small» enough to preserve the stability of (1) when (2) is satisfied. Such a criterion will be obtained in what follows.

In order to be able to speak of the two t-ranges $(d\omega(t) \geqq 0)$, $(d\omega(t) < 0)$, suppose, for instance, that $\omega(t)$ has only a finite number, $N = N(T)$, of local minima on every finite interval $0 \leqq t \leqq T < \infty$. Then $N(T) \to \infty$ as $T \to \infty$, except in the classical case of ultimate monotony, referred to above, a case of assured stability. Except in this case, it follows from (2) that the half-line $0 \leqq t < \infty$ consists of an infinite sequence of closed t-intervals $Q_1, Q_2, \ldots$ and of a complementary infinite sequence of open intervals $R_1, R_2, \ldots$ having the property that $d\omega(t) \geqq 0$ or $d\omega(t) < 0$ holds according as t is in Q or in R, where $Q = \Sigma\, Q_i$ and $R = \Sigma\, R_i$.

It will be proved that (1) *must be stable if, besides* (2),

$$\int_R |d \log \omega(t)| < \infty \tag{3}$$

holds. Actually, the assumption (2) will not be needed in the form in which it stands, since, the portion R of the half-line $0 \leqq t < \infty$ having been taken care of by (3), it will be sufficient to assume that $\omega(t) \to \infty$ holds when t tends to ∞ on the portion Q of the half-line $Q + R$ (provided that $\omega(t)$ is positive throughout). The standard criterion, according to which (1) must be stable if $\omega(t)$ is monotone, will not be assumed; it will appear as a corollary, since (3) is satisfied if R is vacuous.

Suppose that $\omega(t)$ is positive on $Q + R$ and satisfies (2) and (3) on Q and on R respectively. In order to prove that (1) is stable in this case, use will be made of that change of the independent (but *not* of the dependent) variable which occurs in LIOUVILLE's substitution (cf., e. g., [2], pp. 68-70). This means that t will be replaced by s, where $ds = \omega(t)dt$. This $s = s(t)$ is increasing with t, since $\omega(t) > 0$. Hence s tends, as $t \to \infty$, either to a limit $s(\infty) < \infty$ or to $s(\infty) = \infty$. It can be assumed that $s(\infty) = \infty$, since it will be clear that there is no problem if $s(\infty) < \infty$.

If a prime denotes differentiation with respect to s, then substitution of $ds = \omega(t)dt$ in (1) and division of the result by $\omega(t) > 0$ transform (1) into

$$x'' + \varphi(s)x' + x = 0, \tag{4}$$

where $\varphi(s)$ is the function which results if $d \log \omega^2(t)/dt$ is thought of as expressed as a function of s. Thus it is clear from the definition of R that the assumption (3) is equivalent to

$$\int^{\infty} - \varphi^{-}(s)ds < \infty, \qquad (\varphi^{-} \leqq 0), \tag{5}$$

where $\varphi^{-}(s)$ denotes $\varphi(s)$ or 0 according as $\varphi(s) \leqq 0$ or $\varphi(s) > 0$. Accordingly, the assertion, to be proved, is that all solutions $x(s)$ of (4) stay bounded as $s \to \infty$ if (5) is satisfied by $\varphi(s)$. Actually, not only $x(s) = O(1)$ but also $x'(s) = O(1)$ (that is, $x(t) = O(1)$ and $dx(t)/dt = O(\omega(t))$, where $t \to \infty$) will follow.

Consider the following pair of (binary) vectorial differential systems of first order:

$$\text{(6a)} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\varphi(s) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}; \qquad \text{(6b)} \quad \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Clearly, (6a) is equivalent to (4), and (6b) to the case $\varphi = 0$ of (4). Let $A = A(s)$ and $B =$ const. denote the coefficient matrices of (6a) and (6b) respectively, define the matrix $C = C(s)$ by placing

$$(7) \qquad C = X^{-1}(B - A)X, \quad \text{where} \quad X = X(s) = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix},$$

and refer by (C) to the binary differential system which belongs to the matrix $C(s)$ in the same way as the matrix $A(s)$ belongs to (6a).

If the definition (7) of $C(s)$ is compared with LAGRANGE's general rule, concerning the « variation of constants » in homogeneous linear systems, then it is seen that both components of every solution vector of (6a) will stay bounded as $s \to \infty$ if (and only if) the same is true of all solutions of (C). Hence, what must be ascertained is that this will be the case whenever $\varphi(s)$ is subject to (5).

Since $A = A(s)$ and $B =$ const. are the coefficient matrices of (6a) and (6b) respectively, $B - A$ is the diagonal matrix the diagonal elements of which are 0 and $-\varphi$. It follows therefore from (7) that, $X = X(s)$ being an orthogonal matrix, $C = C(s)$ is a symmetric matrix the eigenvalues of which are 0 and $-\varphi(s)$. Hence, if $\varphi^-(s)$ is defined as above, and if $C_s(\xi, \eta)$ denotes the quadratic form belonging to $C(s)$ (at a fixed s), then $C_s(\xi, \eta) \leqq -\varphi^-(s)$ holds whenever $\xi^2 + \eta^2 = 1$. Since the (non-negative) upper bound $-\varphi^-(s)$ of the form C_s (*not* of the absolute value of C_s) is supposed to satisfy (5), the boundedness (as $s \to \infty$) of both components of every solution vector of (C) now follows from the general criterion of [3], p. 558.

REMARK. - It was not used at all that $\omega(t)$ satisfies (2) on Q (that is, on the complement of the set R). In fact, the proof would have remained unaltered if $\omega(t)$ would have been assumed to tend, on Q, to a finite positive limit, instead of to ∞. But since (3) is assumed for the complement ($= R$) of Q, all that can be obtained in this case is contained in a result of OSGOOD ([1]; cf. [2], p. 29).

REFERENCES

[1] W OSGOOD, *On a theorem of oscillation*, « Bulletin of the American Mathematical Society », vol. 25 (1919), pp. 216-221.

[2] G. SANSONE, *Equazioni differenziali nel campo reale*, vol. 2 (2nd ed., 1949).

[3] A. WINTNER, *Asymptotic integration constants*, « American Journal of Mathematics », vol. 68 (1946), pp. 553-559.

[4] — — *Stability and high frequency*, « Journal of Applied Physics », vol. 18 (1947), pp. 941-942.