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On Laguerre and Jacobi polynomials.

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Sunto. - *Using an expansion of* BATEMAN'S, *a number of identities involving the* LAGUERRE *and* JACOBI *polynomials are obtained.*

1. BAILEY [2, p. 65] noted that the formula

$$L_m^{(\alpha)}(z)L_n^{(\alpha)}(z)=\frac{\Gamma(1+\alpha+m)(1+\alpha+n)}{\Gamma(1+\alpha+m+n)} \tag{1.1}$$

$$\sum_{r=0}^{\min(m,n)}\frac{(m+n-2r)!}{r!\,(m-r)!\,(n-r)!\,\Gamma(1+\alpha+r)}z^{2r}L_{m+n-2r}^{(\alpha+2r)}(z)$$

where

$$L_m^{(\alpha)}(z)=\sum_{r=0}^{m}\binom{m+\alpha}{m-r}\frac{(-z)^r}{r!},$$

is a consequence of

$$\sum_0^\infty\frac{z^nL_n^{(\alpha)}(t)}{\Gamma(1+\alpha+n)}=e^z(zt)^{-\alpha/2}J_\alpha\left\{2(zt)^{\frac{1}{2}}\right\} \tag{1.2}$$

and

$$J_\nu(z\cos\theta)J_\nu(z\sin\theta)=\sum_{r=0}^{\infty}\frac{\left(\frac{1}{4}z\sin 2\theta\right)^{\nu+2r}}{r!\,\Gamma(1+\nu+r)}J_{\nu+2r}(z). \tag{1.3}$$

It may be of interest to see what is implied by BATEMANS' s formula [8, p. 370]

$$\frac{1}{2}zJ_\alpha(z\sin\Phi\sin\theta)J_\beta(z\cos\Phi\cos\theta) \tag{1.4}$$

$$=\sin^\alpha\Phi\sin^\alpha\theta\cos^\beta\Phi\cos^\beta\theta\sum_{n=0}^{\infty}(-1)^n(1+\alpha+\beta+2n)J_{1+\alpha+\beta+2n}(z)$$

$$\cdot\frac{\Gamma(1+\alpha+\beta+n)\Gamma(1+\alpha+n)}{n!\,\Gamma^2(1+\alpha)\Gamma(1+\beta+n)}F(-n,\ 1+\alpha+\beta+n,\ 1+\alpha;\ \sin^2\Phi)$$

$$\cdot F(-n,\ 1+\alpha+\beta+1;\ 1+\alpha;\ \sin^2\theta)$$

$$=\sin^\alpha\Phi\sin^\alpha\theta\cos^\alpha\Phi\cos^\beta\theta\sum_{n=0}^{\infty}(-1)^n(1+\alpha+\beta+2n)J_{1+\alpha+\beta+2n}(z)$$

$$\cdot\frac{|n!\,\Gamma(1+\alpha+\beta+n)}{\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}P_n^{(\alpha,\beta)}(\cos 2\Phi)P_n^{(\alpha,\beta)}(\cos 2\theta),$$

where

$$P_n^{(\alpha,\beta)}(x)=\binom{n+\alpha}{\alpha}F\left(-n,\ 1+\alpha+\beta+n;\ 1+\alpha;\ \frac{1-x}{2}\right).$$

It follows from (1.2) that

$$\frac{1}{2}ztJ_\alpha(2zt\sin\Phi\sin\theta)J_\beta(2zt\cos\Phi\cos\theta)$$

$$=\frac{1}{2}e^{-z^2}(zt)^{1+\alpha+\beta}(\sin\Phi\sin\theta)^\alpha(\cos\Phi\cos\theta)^\beta$$

$$\cdot\sum_{m,n=0}^{\infty}\frac{L_m^{(\alpha)}(t^2\sin^2\theta)L_n^{(\beta)}(t^2\cos^2\theta)}{\Gamma(1+\alpha+m)(1+\beta+n)}(z\sin\Phi)^{2m}(z\sin\theta)^{2n}$$

$$=e^{-z^2}(zt)^{1+\alpha+\beta}(\sin\Phi\sin\theta)^\alpha(\cos\Phi\cos\theta)^\beta$$

$$\cdot\sum_{k=0}^{\infty}z^{2k}\sum_{m+n=k}\sin^{2m}\Phi\cos^{2n}\Phi\,\frac{\Gamma_m^{(\alpha)}(t^2\sin^2\theta)L_n^{(\beta)}(t^2\cos^2\theta)}{\Gamma(1+\alpha+m)\Gamma(1+\beta+n)}.$$

On the other hand, the right member of (1.4), with z replaced by $2zt$, becomes

$$(\sin\Phi\sin\theta)^\alpha(\cos\Phi\cos\theta)^\beta\sum_{r=0}^{\infty}(-1)^r(1+\alpha+\beta+2r)\frac{r!\,\Gamma(1+\alpha+\beta+r)}{\Gamma(1+\alpha+r)\Gamma(1+\beta+r)}$$

$$\cdot P_r^{(\alpha,\beta)}(\cos 2\Phi)P_r^{(\alpha,\beta)}(\cos 2\theta)e^{-z^2}(zt)^{1+\alpha+\beta+2r}$$

$$\cdot\sum_{s=0}^{\infty}\frac{L_s^{(1+\alpha+\beta+2r)}(t^2)}{\Gamma(2+\alpha+\beta+2r+s)}z^{2s}.$$

Equating coefficients of z^{2k} we get

$$\text{(1.5)}\qquad \sum_{m+n+k}\sin^{2m}\Phi\cos^{2n}\Phi\,\frac{L_m^{(\alpha)}(t^2\sin^2\theta)L_n^{(\beta)}(t^2\cos^2\theta)}{\Gamma(1+\alpha+m)\Gamma(1+\beta+n)}$$

$$=\sum_{r=0}^{k}(-1)^r(1+\alpha+\beta+2r)\frac{r!\,\Gamma(1+\alpha+\beta+r)}{\Gamma(1+\alpha+r)\Gamma(1+\beta+r)}$$

$$\cdot P_r^{(\alpha,\beta)}(\cos 2\Phi)P_r^{(\alpha,\beta)}(\cos 2\theta)\frac{t^{2r}L_{k-r}^{(1+\alpha+\beta+2r)}(t^2)}{\Gamma(2+\alpha+\beta+k+r)}.$$

If we let $v=t^2\sin^2\theta$, $u=t^2\cos^2\theta$, (1.5) becomes

$$\text{(1.6)}\qquad \sum_{m+n=k}\sin^{2m}\Phi\cos^{2n}\Phi\,\frac{L_m^{(\alpha)}(v)L_n^{(\beta)}(u)}{\Gamma(1+\alpha+m)\Gamma(1+\beta+n)}$$

$$=\sum_{r=0}^{k}(-1)^r(1+\alpha+\beta+2r)\frac{r!\,\Gamma(1+\alpha+\beta+r)}{\Gamma(1+\alpha+r)\Gamma(1+\beta+r)}$$

$$\cdot P_r^{(\alpha,\beta)}(\cos 2\Phi)P_r^{(\alpha,\beta)}\left(\frac{u-v}{u+v}\right)(u+v)^r\frac{L_{k-r}^{(1+\alpha+\beta+2r)}(u+v)}{\Gamma(2+\alpha+\beta+k+r)},$$

or if we prefer

$$(1.7)\qquad \sum_{m+n=k} y^m x^n \frac{L_m^{(\alpha)}(v)L_n^{(\beta)}(u)}{\Gamma(1+\alpha+m)\Gamma(1+\beta+n)}$$

$$= \sum_{r=0}^{k} (-1)^r(1+\alpha+\beta+2r \frac{r!\,\Gamma(1+\alpha+\beta+r)}{\Gamma(1+\alpha+r)\Gamma(1+\beta+r)}$$

$$\cdot (x+y)^k(u+v)^r P_r^{(\alpha,\beta)}\left(\frac{x-y}{x+y}\right) P_r^{(\alpha,\beta)}\left(\frac{u-v}{u+v}\right) \frac{L_{k-r}^{(1+\alpha+\beta+2r)}(u+v)}{\Gamma(2+\alpha+\beta+k+r)}.$$

For $\alpha=\beta=0$ this reduces to

$$\sum_{m+n=k} y^m x^n L_m(v)L_n(u)/m!\,n!$$

$$= \sum_{r=0}^{k} (-1)^r(1+2r)(x+y)^k(u+v)^r P_r\left(\frac{x-y}{x+y}\right) P_r\left(\frac{u-v}{u+v}\right) \frac{L_{k-r}^{(1+2r)}(u+v)}{(1+k+r)!}.$$

In (1.7) take $y=1$, $x=-1$ and we find that

$$(1.8)\qquad \sum_{m+n=k} (-1)^n \frac{L_m^{(\alpha)}(v)L_n^{(\beta)}(u)}{\Gamma(1+\alpha m)\Gamma(1+\beta+n)} = \frac{(u+v)^k P_k^{(\alpha,\beta)}\left(\dfrac{u-v}{u+v}\right)}{\Gamma(1+\alpha+k)\Gamma(1+\beta+k)}.$$

In particular for $u=(1+x)/2$, $v=(1-x)/2$, (1.8) reduces to

$$P_k^{(\alpha,\beta)}(x) = \sum_{m+n=k} (-1)^n \frac{\Gamma(1+\alpha+k)\Gamma(1+\beta+k)}{\Gamma(1+\alpha+n)\Gamma(1+\beta+n)} L_m^{(\beta)}\left(\frac{1+x}{2}\right) L_n^{(\alpha)}\left(\frac{1-x}{2}\right),$$

a formula proved by RAINVILLE [7].

2. It is not difficult to give a direct proof of (1.8). Indeed we have

$$\sum_{m+n=k} y^m x^n \frac{L_m^{(\alpha)}(v)L_n^{(\beta)}(u)}{\Gamma(1+\alpha+m)\Gamma(1+\beta+n)}$$

$$= \sum_{m+n=k} y^m x^n \sum_{r=0}^{n} (-1)^r \frac{u^r}{(n-r)!r!\Gamma(1+\beta+r)} \sum_{s=0}^{m} (-1)^s \frac{v^s}{(m-s)!s!\Gamma(1+\alpha+s)}$$

$$= \sum_{r+s\le k} (-1)^{r+s} \frac{u^r v^s}{r!\,s!\,\Gamma(1+\beta+r)\Gamma(1+\alpha+s} \sum_{m+n=k} \frac{y^m y^n}{(n-r)!\,(m-s)!}$$

$$= \sum_{r+s\le k} (-1)^{r+s} \frac{(xu)^r(yv)^s(k+y)^{k-r-s}}{r!\,s!\,(k-r-s)!\,\Gamma(1+\beta+r)\Gamma(1+\alpha+s)}$$

$$= \sum_{m=0}^{k} (-1)^m \frac{(x+y)^{k-m}}{(k-m)!} \sum_{r+s=m} \frac{(xy)^r(yv)^s}{r!\,s!\,\Gamma(1+r+\beta)\Gamma(1+s+\beta)}$$

$$= \sum_{m=0} (-1)^m \frac{(xu-yv)^m(x+y)^{k-m}}{(k-m)!\,\Gamma(1+\alpha+m)\Gamma(1+\beta+m)} P_m^{(\alpha,\beta)}\left(\frac{xu+yv}{xu-yv}\right).$$

For $y=1$, $x=-1$, this identity reduces to (1.8). Comparison with (1.7) yields

$$(2.1)\quad \sum_{m=0}^{k}(-1)^m \frac{(xu-yv)^m(x+y)^{k-m}}{(k-m)!\,\Gamma(1+\alpha+m)\Gamma(1+\beta+m)} P_m^{(\alpha,\beta)}\left(\frac{xu+yv}{xu-yv}\right)$$

$$=\sum_{r=0}^{k}(-1)^r(1+\alpha+\beta+2r)\frac{r!\,\Gamma(1+\beta+r)}{\Gamma(1+\alpha+r)\Gamma(1+\beta+r)}$$

$$\cdot(x+y)^k(u+v)^r P_r^{(\alpha,\beta)}\left(\frac{x-y}{x+y}\right)P_r^{(\alpha,\beta)}\left(\frac{u-v}{u+v}\right)\frac{L_{k-r}^{(1+\alpha+\beta+2r)}(u+v)}{\Gamma(2+\alpha+\beta+k+r)}.$$

Replace x, y, u, v by

$$\frac{1}{2}(1+x)z,\ \ \frac{1}{2}(1-x)z,\ \ \frac{1}{2}(1+y)z,\ \ \frac{1}{2}(1-y)z,$$

respectively, then (2.1) become

$$(2.2)\quad \sum_{m=0}^{k}(-1)^m \frac{(x+y)^m z^m}{2^m(k+m)!\,\Gamma(1+\alpha+m)\Gamma(1+\beta+m)} P_m^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right)$$

$$=\sum_{r=0}^{k}(-1^r(1+\alpha+\beta+2r)\frac{r!\,\Gamma(1+\alpha+\beta+r)}{\Gamma(1+\alpha+r)\Gamma(1+\beta+r)}$$

$$\cdot z^r P_r^{(\alpha,\beta)}(x)P_r^{)\alpha,\beta)}(y)\,\frac{L_{k-r}^{(1+\alpha+\beta+2r)}(z)}{\Gamma(2+\alpha+\beta+k+r)}.$$

Comparing coefficients of z^k in both members of (2.2) we get

$$(2.3)\quad \left(\frac{x+y}{2}\right)^k P_k^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right)=\sum_{r=0}^{k}\frac{(1+\alpha+\beta+2r)\,r!\,\Gamma(1+\alpha+\beta+r)}{(k-r)!\,\Gamma(2+\alpha+\beta+k+r)}$$

$$\cdot\frac{\Gamma(1+\alpha+k)\Gamma(1+\beta+k)}{\Gamma(1+\alpha+r)\Gamma(1+\beta+r)}P_r^{(\alpha,\beta)}(x)P_r^{(\alpha,\beta)}(y).$$

Next multiply both sides of (2.2) by $e^{-z}z^{\alpha+\beta+k}$ and integrate from 0 to ∞. Since

$$\int_0^\infty e^{-z}z^{\alpha+\beta+k+r}L_{k-r}^{(1+\alpha+\beta+2r)}(z)dz=0 \qquad (k>r),$$

$$\int_0^\infty e^{-z}z^{\alpha+\beta+k+r}dz=\Gamma(1+\alpha+\beta+k+r) \qquad (\alpha+\beta+k+r>-1),$$

(2.2) yields

$$(2.4)\quad P_k^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(y)=\frac{(1+\alpha)_k(1+\beta)_k}{k!}$$

$$\cdot\sum_{m=0}^{k}(-1)^{k-m}\frac{(1+\alpha+\beta+k)_m(x+y)^m}{2^m(k-m)!\,(1+\alpha)_m(1+\beta)_m}P_m^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right).$$

Since $P_m^{(\alpha,\beta)}(x)$ is a polynomial in α, β (as well as x), (2.4) holds for all values of the parameters.

The formulas (2.3) and (2.4) are due to BATEMAN and are stated in [5, p. 392]; (2.3) was proved in [4]. (2.4) was rediscovered by BAILEY [3] and BRAFMAN [6].

If in (2.1) we take $y = 0$ and change the notation slightly we find that

$$(2.5)\quad L_k^{(\beta)}\left(\frac{1}{2}x(1+y)\right) = \sum_{r=0}^{k} (-1)^r(1+\alpha+\beta+2r\frac{(1+\beta+r)_{k-r}}{(1+\alpha+\beta+r)_{k+1}} \cdot x^r P_r^{(\alpha,\beta)}(y) L_{k-r}^{(1+\alpha+\beta+2r)}(x);$$

in particular for $y = 1$, (2.5) reduces to

$$(2.6)\quad L_k^{(\beta)}(x) = \sum_{r=0}^{k} (-1)^r(-1)^r(1+\alpha+\beta+2r)\frac{(1+\alpha)_r(1+\beta+r)_{k-r}}{r!\,(1+\alpha+\beta+r)_{k+1}} \cdot x^r L_{k-r}^{(1+\alpha+\beta+2r)}(x).$$

3. If

$$f(x) = \sum_{r=0}^{k} a_r x^r (1-x)^{k-r},$$

where a_r is independent of x, then it is easily verified that

$$(3.1)\qquad a_m = \sum_{r=0}^{m} \binom{k-r}{m-r} \frac{f^{(r)}(0)}{r!}.$$

In (1.5) take $x = \sin^2 \Phi$; then (3.1) yields

$$(3.2)\quad \frac{L_m^{(\alpha)}(t^2\sin^2\theta)L_n^{(\beta)}(t^2\cos^2\theta)}{\Gamma(1+\alpha+m)\Gamma(1+\beta+n)} = \binom{m+n}{n} \sum_{r=0}^{m+n} (-1)^r(1+\alpha+\beta+2r)\frac{(1+\alpha+\beta+r)}{\Gamma(1+\alpha)\Gamma(1+\beta+r)} \cdot {}_3F_2\left[\begin{matrix}-r,\ -m,\ 1+\alpha+\beta+r \\ -m-n,\ 1+\alpha\end{matrix}\right] P_r^{(\alpha,\beta)}(\cos 2\theta) \frac{t^{2r} L_{m+n-r}^{(1+\alpha+\beta+2r)}(t^2)}{\Gamma(2+\alpha+\beta+m+n+r)},$$

or

$$(3.3)\quad \frac{L_m^{(\alpha)}(v)\Gamma_n^{(\beta)}(u)}{\Gamma(1+\alpha+m)\Gamma(1+\beta+n)} = \binom{m+n}{m} \sum_{r=0}^{m+n} (-1)^r(1+\alpha+\beta+2r)\frac{\Gamma(1+\alpha+\beta+r)}{\Gamma(1+\alpha)\Gamma(1+\beta+r)} \cdot {}_3F_2\left[\begin{matrix}-r,\ -m,\ 1+\alpha+\beta+r \\ -m\,n,\ 1+\alpha)\end{matrix}\right] (u+v)^r P_r^{(\alpha,\beta)}\left(\frac{u-v}{u+v}\right) \frac{L_{m+u-r}^{(1+\alpha+\beta+2r)}(u+v)}{\Gamma(2+\alpha+\beta+m+n+r)}.$$

In particular for $v = 0$, (3.3) becomes

$$\frac{L_n(u)}{m!\,\Gamma(1+\beta+n)} = \binom{m+n}{m} \sum_{r=0}^{m+n} (-1)^r (1+\alpha+\beta+2r) \frac{(1+\alpha+\beta+r)}{\Gamma(1+\beta+r)}$$

$$\cdot \binom{\alpha+r}{r} \cdot {}_3F_2 \left[\begin{matrix} -r, & -m, & 1+\alpha+\beta+r \\ & -m-n, & 1+\alpha \end{matrix} \right] \frac{u^r L_{m+n-r}^{(1+\alpha+\beta+2r)}(u)}{\Gamma(2+\alpha+\beta+m+n+r)}.$$

which may be written in the form

$$(3.4) \qquad L_n^{(\beta)}(u) = \frac{(m+n)!}{n!} \sum_{r=0}^{m+n} (-1)^r (1+\alpha+\beta+2r) \frac{(1+\alpha)_r (1+\beta+r)_{n-r}}{r!\,(1+\alpha+\beta+r)_{m+n+1}}$$

$$\cdot {}_3F_2 \left[\begin{matrix} -r, & -m, & 1+\alpha+\beta+r \\ & -m-n, & 1+\alpha \end{matrix} \right] u^r L_{m+n-r}^{(1+\alpha+\beta+2r)}(u).$$

For $m = 0$, (3.4) evidently reduces to (2.6).

When $m = n$, $\alpha = \beta$, we have [1, p. 16, § 3.3]

$${}_3F_2 \left[\begin{matrix} -r, & 1+2\alpha+r, & -m \\ & 1+\alpha, & -2m \end{matrix} \right] =$$

$$= \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2} - m\right) \Gamma(1+\alpha) \Gamma(-\alpha-m)}{\Gamma\left(\frac{1}{2} - \frac{r}{2}\right) \Gamma\left(1+\alpha+\frac{r}{2}\right) \Gamma\left(\frac{1}{2} + \frac{r}{2} - m\right) \Gamma\left(-\alpha - \frac{r}{2} - m\right)}$$

$$= \frac{\left(\frac{1}{2}\right)_s (1+\alpha+m)_s}{(1+\alpha)_s \left(\frac{1}{2} - m\right)_s} \qquad (r = 2s)$$

when r is odd the series vanishes). Thus in the special case (3.3) reduces to

$$(3.5 \qquad L_m^{(\alpha)}(u) L_m^{(\alpha)}(v) =$$

$$= \binom{2m}{m} \sum_{s=0}^{m} (1+2\alpha+4s) \frac{(1+\alpha+s)_{m-s} (1+\alpha+2s)_{m-s}}{(1+2\alpha+2s)_{2m+1}} \frac{\left(\frac{1}{2}\right)_s}{\left(\frac{1}{2} - m\right)}$$

$$\cdot (u+v)^{2s} P_{2s}^{(\alpha,\alpha)} \left(\frac{u-v}{u+v} \right) L_{2m-2s}^{(1+2\alpha+4s)}(u+v).$$

In this connection we may recall the simpler formula of BAILEY [2, (1.1)]:

$$L_m^{(\alpha)}(u)L_m^{(\alpha)}(v) = \frac{\Gamma(1+\alpha+m)}{m!}\sum_{r=0}^{m}\frac{(uv)^r L_{m-r}^{(\alpha+2r)}(u+v)}{r!\,\Gamma(1+\alpha+r)}.$$

4. If in (2.3) we take $y = -x$ we get

$$(4.1)\qquad \frac{(1+\alpha+\beta+k)_k}{k!}(1-x^2)^k$$

$$= \sum_{r=0}^{k}\frac{(1+\alpha+\beta+2r)r!\,\Gamma(1+\alpha+\beta+r)}{(k-r)!\,\Gamma(2+\alpha+\beta+k+r)}$$

$$\cdot\frac{\Gamma(1+\alpha+k)\Gamma(1+\beta+k)}{\Gamma(1+\alpha+r)\Gamma(1+\beta+r)}P_r^{(\alpha,\beta)}(x)P_r^{(\alpha,\beta)}(-x).$$

Similarly (2.4) yields

$$(4.2)\qquad P_k^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(-x) = \frac{(1+\alpha)_k(1+\beta)_k}{k!}$$

$$\cdot\sum_{m=0}^{k}(-1)^{k-m}\frac{(1+\alpha+\beta+k)_m(1+\alpha+\beta+m)_m}{2^m m!\,(k-m)!\,(1+\alpha)_m(1+\beta)_m}(1-x^2)^m.$$

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