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Arthur E. Danese

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Zanichelli

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On a theorem of Merli concerning ultraspherical polynomials.

by ARTHUR E. DANESE (a Rochester, New York, U.S.A.)

Summary. - *The range of validity of a theorem of Merli' s is extended.*

Let $P_n^\lambda(x)$ denote the ultraspherical polynomial of degree n and order λ as defined in SZEGÖ [1]. Consider $\Delta_n^\lambda(x, k) = [F_n^\lambda(x)]^2 - k \cdot F_{n-1}^\lambda(x) F_{n+1}^\lambda(x)$, where $F_n^\lambda(x) = P_n^\lambda(x)/P_n^\lambda(1)$ and k is a real number. MERLI [2] proves the following theorem:

If $n \geq 1$, $1 < \lambda < 2$, $-\infty < x < \infty$, $k \leq \dfrac{n(n+\lambda+1)}{n(n+\lambda+1)+1}$, *then* $\Delta_n^\lambda(x, k) > 0$.

In this note we improve on this result by extending the range of λ and k.

THEOREM 1. - *If* $\lambda > 0$, $n \geq 1$, *then*

(1) $\quad |x| \leq 1,\ 0 \leq k \leq 1$ *implies* $\Delta_n^\lambda(x, k) \geq 0$,

(2) $\quad |x| > 1,\ k < \dfrac{(n+2\lambda)(n+\lambda-1)}{(n+\lambda)(n+2\lambda-1)}$ *implies* $\Delta_n^\lambda(x, k) < 0$,

(3) $\quad |x| > 1,\ k \geq 1$ *implies* $\Delta_n^\lambda(x, k) < 0$.

In (2) *the function in* λ *and* n *is the best possible one which will yield the inequality.*

Proof. (1) Since $\Delta_n^\lambda(x, 1) \geq 0$ (see (a) below) and $\Delta_n^\lambda(x, 0) \geq 0$, the result is immediate.

(2) and (3) Consider the function $g(x) = \dfrac{F_{n+1}^\lambda(x) F_{n-1}^\lambda(x)}{[F_n^\lambda(x)]^2}$, defined for $|x| \geq 1$.

([1]) SZEGÖ G., *Orthogonal Polynomials*, New York, 1939.

([2]) MERLI L, *Sopra alcune disuguaglianze riguardanti i polinomi ultrasferici di Jacobi*, Atti del IV Congresso dell' Unione Matematica, vol. II (1951), pp. 151-155.

Then $g(1) = 1$ and $\lim_{x \to \infty} g(x) = \frac{(n+2\lambda)(n+\lambda-1)}{(n+\lambda)(n+2\lambda-1)} < 1$ (3).

Next we show that $g(x)$ is non-increasing in $|x| \geq 1$. The author (4) has shown that

$$(a) \quad \Delta_n^\lambda(x, 1) = \frac{n!\, n-1)!\, 4\lambda(1-x^2)}{\Gamma(n+2\lambda)\Gamma(n+2\lambda+1)} \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} \frac{(i+\lambda)(2\lambda)_j[\Gamma(i+2\lambda)]^2}{(j+1)!\,(2\lambda)_i\, i!} [F_i^\lambda(x)]^2, \quad n \geq 1.$$

Hence, for $x \geq 1$, $\lambda > 0$, $\frac{d}{dx}\Delta_n^\lambda(x, 1) < 0$ since $F_i^\lambda(x) \cdot \frac{d}{dx} F_i^\lambda(x) > 0$.

Therefore, $\frac{\Delta_n^\lambda(x, 1)}{F_{n+1}(x) F_{n-1}(x)} = g(x) - 1$ is non-increasing for $x \geq 1$, $\lambda > 0$ and also for $x < -1$, $\lambda > 0$, by symmetry.

The results now follow immediately.

One can prove similarly:

THEOREM 2. - *If* $\lambda > 0$, $n \geq 1$ *and* $\delta_n^\lambda(x, k) = [P_n^\lambda(x)]^2 - kP_{n+1}^\lambda(x) P_{n-1}^\lambda(x)$, *then*

(1) $|x| \leq 1$ *implies* $\delta_n^\lambda(x, k) \geq 0$, $0 \leq k \leq 1$,

(2) $|x| > 1$, $k < \frac{(n+1)(n+\lambda-1)}{n(n+\lambda)}$ *implies* $\delta_n^\lambda(x, k) > 0$,

(3) $|x| > 1$, $k < \frac{(n+1)(n+2\lambda-1)}{n(n+2\lambda)}$ *implies* $\delta_n^\lambda(x, k) < 0$.

(3) SZEGÖ G., loc cit., 4 7.3 and 4.7.31.

(4) DANESE A. E., *Explicit evaluations of Turán expressions,* « Annali di Matematica Pura ed Applicata », vol. 38 (1955), p. 344.