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Problemi diofantei.

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Zanichelli

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Problemi diofantei.

Nota di Alfredo Moessner (a Gunzenhausen)

Sunto. - *Si dànno alcune relazioni numeriche di natura diofantea.*

I. Se $M + P = Q$, allora si ha

$$\left\{\begin{aligned} & M^2 + P^2 + Q^2 = \frac{1}{3} \cdot [(P - M)^2 + (M + Q)^2 + (P + Q)^2] = 2 \cdot (M^2 + PQ) \\ & M^4 + P^4 + Q^4 = \frac{1}{3^3} \cdot [(P - M)^4 + (M + Q)^4 + (P + Q)^4] = 2 \cdot (M^2 + PQ)^2 \\ & (M^2 + P^2 + Q^2)^2 = 2(M^4 + P^4 + Q^4) \end{aligned}\right.$$

Esempio. - Sia $M = 1$, $P = 5$, $Q = 6$; allora

$$\left\{\begin{aligned} & 1^2 + 5^2 + 6^2 = \frac{1}{3} \cdot [4^2 + 7^2 + 11^2] = 2 \cdot (1^2 + 5 \quad 6) \\ & 1^4 + 5^4 + 6^4 = \frac{1}{3^2} \cdot [4^4 + 7^4 + 11^4] = 2 \cdot (1^2 + 5 \cdot 6)^2 \\ & (1^2 + 5^2 + 6^2)^2 = 2(1^4 + 5^4 + 6^4) = 62^2. \end{aligned}\right.$$

II. Se $A^n + B^n = C^n$, e $ABC = D$, allora si ha

$$\left\{\begin{aligned}
&-A^{3n} - B^{3n} + C^{3n} + 3(2D^n) = 3D^n + (3D^n - C^{3n}) +\\
&\qquad\qquad + (3D^n + A^{3n}) + (3D^n + B^{3n})\\
&(-A)^{6n} + (-B)^{6n} + C^{6n} + 3(2D^n)^2 = 3D^{2n} + (3D^n - C^{3n})^2 +\\
&\qquad\qquad + (3D^n + A^{3n})^2 + (3D^n + B^{3n})^2\\
&-A^{9n} - B^{9n} + C^{6n} + 3(2D^n)^3 = 3D^{3n} + (3D^n - C^{3n})^3 +\\
&\qquad\qquad + (3D^n + A^{3n}) + (3D^n + B^{3n})^3\\
&(-A)^{12n} + (-B)^{12n} + C^{12n} + 3(2D^n)^4 = 3D^n + (3D^n - C^{3n})^4 +\\
&\qquad\qquad + (3D^n + A^{3n})^4 + (3D^n + B^{3n})^4.
\end{aligned}\right.$$

III. Se $A^2 + B^2 = C^2$, per II si ha

$$\left\{\begin{aligned}
&-A^6 - B^6 + C^6 + 3(2D^2) = 3D^2 + (3D^2 + A^6) + (3D^2 + B^6) + (3D^2 - C^6)\\
&A^{12} + B^{12} + C^{12} + 3(2D^2)^2 = 3D^4 + (3D^2 + A^6)^2 + (3D^2 + B^6)^2 + (3D^2 - C^6)^2\\
&-A^{18} - B^{18} + C^{18} + 3(2D^2)^3 = 3D^6 + (3D^2 + A^6)^3 + (3D^2 + B^6)^3 + (3D^2 - C^6)^3\\
&A^{24} + B^{24} + C^{24} + 3(2D^2)^4 = 3D^8 + (3D^2 + A^6)^4 + (3D^2 + B^6)^4 + (3D^2 - C^6)^4.
\end{aligned}\right.$$

Esempio. - Sia $A = 3$, $B = 4$, $C = 5$, $D = 3 \cdot 4 \cdot 5 = 60$; allora si ha

$$\left\{\begin{aligned}
&-3^6 - 4^6 + 5^6 + 3 \cdot (2 \cdot 60^2) = 3 \cdot 60^2 + 11529 + 14896 + (-4825)\\
&3^{12} + 4^{12} + 5^{12} + 3 \cdot (2 \cdot 60^2)^2 = 3 \cdot 60^4 + 11529^2 + 14896^2 + (-4825)^2\\
&-3^{18} - 4^{18} + 5^{18} + 3 \cdot (2 \cdot 60^2)^3 = 3 \cdot 60^6 + 11529^3 + 14896^3 + (-4825)^3\\
&3^{24} + 4^{24} + 5^{24} + 3 \cdot (2 \cdot 60^2)^4 = 3 \cdot 60^8 + 11529^4 + 14896^4 + (-4825^4.
\end{aligned}\right.$$

IV. Sia $A^n + B^n = C^n$ e $ABC = D$, allora si ha

$$\left\{\begin{aligned}
&-A^{3n} - B^{3n} + C^{3n} = D^n + D^n + D^n\\
&(-A)^{3n} \cdot (-B)^{3n} \cdot C^{3n} = D^n \cdot D^n \cdot D^n.
\end{aligned}\right.$$

Esempio. - Sia $A = 3$, $B = 4$, $C = 5$, $D = 3 \cdot 4 \cdot 5 = 60$, $n = 2$, allora si ha

$$\left\{\begin{aligned}
&-3^6 - 4^6 + 5^6 = 60^2 + 60^2 + 60^2\\
&(-3)^6 \cdot (-4)^6 \cdot 5^6 = 60^2 \cdot 60^2 \cdot 60^2.
\end{aligned}\right.$$

V. Se $A^2 + B^2 = C^2$; $AB = U$, $AC = V$, $BC = W$, allora si ha

$$\left\{\begin{array}{llll}
A^4 + W^2 = & B^4 + V^2 = & C^4 - U^2 = & -U^2 + V^2 + W^2 = \dfrac{A^4 + B^4 + C^4}{2}\\
W^4 & + V^4 & + U^4 = & \left(\dfrac{A^4 + B^4 + C^4}{2}\right)^2\\
2(W^4 & + V^4 & + U^4) = & A^8 + B^8 + C^8\\
A \cdot W = & B \cdot V = & C \cdot U = & ABC
\end{array}\right.$$

ESEMPIO. - Per $A=3$, $B=4$, $C=5$, $U=3\cdot4=12$, $V=3\cdot5=15$, $W=4\cdot5=20$ si ha

$$\left\{\begin{array}{llll} 3^4+20^2=4^4+15^2=5^4-12^2=-12^2+15^2+20^2= & & & \dfrac{3^4+4^4+5^4}{2} \\ 20^4 \quad +15^4 \quad +12^4= & & & \left(\dfrac{3^4+4^4+5^4}{2}\right)^2 \\ 2(20^4 \quad +15^4 \quad +12^4)= & & & 3^8+4^8+5^8 \\ 3\cdot20= \quad 4\cdot15= \quad 5\cdot12= & & & 3\cdot4\cdot5=60. \end{array}\right.$$