Otto Szász

On the relative extrema of ultraspherical polynomials

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **5** (1950), n.2, p. 125–127.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1950_3_5_2_125_0>

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On the relative extrema of ultraspherical polynomials.

Nota di OTTO SZÁSZ (a Los Angeles, Calif.) (1).

Summary. - *A result of* SZEGÖ *on the relative extrema of* LEGENDRE *polynomials is here generalized to ultraspherical polynomials.*

1. The ultraspherical polynomial $P_n^{\lambda}(x)$ has n zeros in the interval $-1 < x < 1$, hence $n-1$ relative extrema in that interval.

(1) The preparation of this paper was sponsored (in part) by the Office of Naval Research.

Denote by $\mu_{r,n}(\lambda)$, $r = 1, 2, \ldots, n-1$ the successive relative extrema of $|P_n^\lambda(x)|$, when x decreases from 1 to -1. We assume $\lambda > 0$; writing μ for $\mu(\lambda)$, we have [2]

$$1 > \mu_{1,n} > \mu_{2,n} > \ldots > \mu_{h,n}, \qquad h = \left[\frac{n}{2}\right], \qquad n \geq 2;$$

for reasons of symmetry the second half of the μ-sequence is increasing.

In the preceding note SZEGÖ proved for the LEGENDRE polynomials $\left(\lambda = \frac{1}{2}\right)$ the interesting inequalities

$$\mu_{r,n} > \mu_{r,n+1}, \qquad n = r+1,\ r+2, \ldots. \tag{1.1}$$

We show here that his elegant procedure can be extended to the ultraspherical polynomials. The result is

$$\frac{\Gamma(n+1)}{\Gamma(n+2\lambda)}\mu_{r,n}(\lambda) > \frac{\Gamma(n+2)}{\Gamma(n+2\lambda+1)}\mu_{r,n+1}, \qquad \lambda > 0, \qquad n \geq r+1; \tag{1.2}$$

that is, the sequence $\rho_{r,n}(\lambda) = \frac{\Gamma(n+1)}{\Gamma(n+2\lambda)}\mu_{r,n}(\lambda)$ is monotone in n.

2. Let

$$x_{1,n} > x_{2,n} > \ldots > x_{n-1,n} \tag{2.1}$$

be the abscissas which correspond to the extrema of $P^\lambda_n(x)$; thus (2.1) are the zeros of $\frac{d}{dx}P_n^\lambda(x)$. In view of (cf. [2], p. 81]

$$\frac{d}{dx}P_n^\lambda(x) = 2\lambda P_{n-1}^{\lambda+1}(x), \tag{2.2}$$

the $x_{r,n}$ are the zeros of $P_{n-1}^{\lambda+1}(x)$; accordingly

$$x_{1,n+1} > x_{2,n+1} > \ldots > x_{n,n+1}$$

are the zeros of $P_n^{\lambda+1}(x)$. We have [cf. [2], p. 45]

$$x_{1,n+1} > x_{1,n} > x_{2,n+1} > \ldots > x_{n-1,n+1} > x_{n-1,n} > x_{n,n+1},$$

thus the polynomials $P_{n-1}^{\lambda+1}(x)$ and $P_n^{\lambda+1}(x)$ are of opposite sign in the intervals

$$x_{r,n} < x < x_{r,n+1}, \qquad r = 1,\ 2, \ldots,\ n-1.$$

[2] Cf. G. SZEGÖ, *Orthogonal polynomials*, New York, 1939, p. 163-164

3. We use the identities [cf. ([2]), p. 83]

$$2\lambda(1-x^2)P_{n-1}^{\lambda+1}(x) \begin{cases} =(n+2\lambda-1)P_{n-1}^{\lambda}(x)-nxP_n^{\lambda}(x) \\ =(n+2\lambda)xP_n^{\lambda}(x)-(n+1)P_{n+1}^{\lambda}(x), \end{cases}$$

It follows that

$$2\lambda(1-x^2)P_n^{\lambda+1}(x)=(n+2\lambda)P_n^{\lambda}(x)-(n+1)xP_{n+1}^{\lambda}(x),$$

and

$$2\lambda(1-x)\{P_{n-1}^{\lambda+1}(x)+P_n^{\lambda+1}(x)\}=(n+2\lambda)P_n^{\lambda}(x)-(n+1)P_{n+1}^{\lambda}(x).$$

Replacing x by $-x$, we find, in view of $P_n^{\lambda}(-x)=(-1)^nP_n^{\lambda}(x)$

$$2\lambda(1+x)\{P_n^{\lambda+1}(x)-P_{n-1}^{\lambda+1}(x)\}=(n+2\lambda)P_n^{\lambda}(x)+(n+1)P_{n+1}^{\lambda}(x),$$

thus

$$4\lambda^2(1-x^2)\{(P_n^{\lambda+1}(x))^2-(P_{n-1}^{\lambda+1}(x))^2\}=(n+2\lambda)^2(P_n^{\lambda}(x))^2-(n+1)^2(P_{n+1}^{\lambda}(x))^2.$$

For $x=x_{r,n+1}$ we now get

$$(n+1)^2(P_{n+1}^{\lambda}(x_{r,n+1}))^2=(n+1)^2\mu^2_{r,n+1}=$$
$$=[(n+2\lambda)^2(P_n^{\lambda}(x))^2+4\lambda^2(1-x^2)(P_{n-1}^{\lambda+1}(x))^2]_{x=x_{r,n+1}}=g(x_{r,n+1}),$$

where

$$g(x)=(n+2\lambda)^2(P_n^{\lambda}(x))^2+4\lambda^2(1-x^2)(P_{n-1}^{\lambda+1}(x))^2.$$

Now, in view of (2.2)

$$g'(x)=4\lambda P_{n-1}^{\lambda+1}(x)\left\{(n+2\lambda)^2P_n^{\lambda}(x)-x\frac{d}{dx}(P_n^{\lambda}(x))+(1-x^2)\frac{d^2}{dx^2}(P_n^{\lambda}(x))\right\}.$$

We employ the formulas [cf. ([2]), pp. 80 and 84]

$$(1-x^2)\frac{d^2}{dx^2}(P_n^{\lambda}(x))=(2\lambda+1)x\frac{d}{dx}(P_n^{\lambda}(x))-n(n+2\lambda)P_n^{\lambda}(x),$$

and

$$(n+2\lambda)P_n^{\lambda}(x)=\frac{d}{dx}(P_{n+1}^{\lambda}(x))=x\frac{d}{dx}(P_n^{\lambda}(x));$$

we then find

$$g'(x)=8\lambda^2P_{n-1}^{\lambda+1}(x)\left\{x\frac{d}{dx}P_n^{\lambda}(x)+(n+2\lambda)P_n^{\lambda}(x)\right\}=$$
$$=8\lambda^2P_{n-1}^{\lambda+1}(x)\frac{d}{dx}(P_{n+1}^{\lambda}(x))=16\lambda^3P_{n-1}^{\lambda+1}(x)P_n^{\lambda+1}(x).$$

It follows that $g'(x)<0$ in the interval $x_{r,n}<x<x_{r,n+1}$, hence $g(x)$ is decreasing in this interval. Thus

$$g(x_{r,n+1})=(n+1)^2\mu^2_{r,n+1}<g(x_{r,n})=(n+2\lambda)^2(P_n^{\lambda}(x_{r,n}))^2=$$
$$=(n+2\lambda)^2\mu^2_{r,n},$$

from which (1.2) follows.

Observe that $\Gamma(2\lambda)\rho_{r,n}(\lambda)$ are the extrema of the normalized polynomial $P_n^{\lambda}(x)/P_n^{\lambda}(1)$.