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On the relative extrema of Legendre polynomials

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On the relative extrema of Legendre Polynomials.

Nota di G. SZEGÖ (a Stanford University, Calif.).

Summary. - *The relative maxima of* $|P_n(x)|$ *are decreasing if the degree* n *increases.*

1. For $n \geqq 2$, LEGENDRE's polynomial $P_n(x)$ is an oscillating function. Let

(1) $$\mu_{1,n},\ \mu_{2,n},\ \dots,\ \mu_{n-1,n}$$

denote the successive relative maxima of $|P_n(x)|$ when x decreases from 1 to - 1. *The first half of this sequence is decreasing* [1]:

(2) $$1 > \mu_{1,n} > \mu_{2,n} > \dots > \mu_{h,n}, \qquad h = [n/2].$$

For reasons of symmetry the second half of the sequence (1) is increasing.

It is interesting to compare the maxima $\mu_{r,n}$ for different values of n. We prove in this Note that for a fixed r, *the maxima* $\mu_{r,n}$ *are decreasing when* n *increases*, $n \geqq r + 1$; i. e.

(3) $$\mu_{r,n} > \mu_{r,n+1}, \qquad n = r+1,\ r+2, \dots$$ [2].

2. Let

(4) $$x_{1,n} > x_{2,n} > \dots > x_{n-1,n}$$

be the abscissas to which the extrema (1) of $P_n(x)$ correspond. The numbers

(5) $$x_{1,n+1} > x_{2,n+1} > \dots > x_{n,n+1}$$

(1) Cf. G. SZEGÖ, *Orthogonal polynomials*, New York, 1939, p. 159.

(2) This inequality was communicated to me as a conjecture by Dr. JOHN TODD. A proof of (3) for fixed r and *large* n was supplied by Dr. R. COOPER.

separate the numbers (4). This follows, for instance, from the fact that (4) and (5) are the zeros of $P'_n(x)$ and $P'_{n+1}(x)$ which are orthogonal polynomials associated with the same weight function (namely $1-x^2$ in $-1, 1$). We conclude that the polynomials $P'_n(x)$ and $P'_{n+1}(x)$ are of opposite sign in the intervals

$$x_{r,n} < x < x_{r,n+1}, \qquad r = 1, 2, \ldots, n-1. \tag{6}$$

3. Now we use the identity (SZEGÖ, loc. cit., p. 370):

$$(1-x)(P'_{n+1}(x) + P'_n(x)) = (n+1)(P_n(x) - P_{n+1}(x)). \tag{7}$$

Replacing x by $-x$ we find

$$(1+x)(P'_{n+1}(x) - P'_n(x)) = (n+1)(P_n(x) + P_{n+1}(x)). \tag{8}$$

so that

$$(1-x^2)((P'_{n+1}(x))^2 - (P'_n(x))^2) = (n+1)^2((P_n(x))^2 - (P_{n+1}(x))^2. \tag{9}$$

Substituting $x = x_{r,n+1}$

$$\mu^2_{r,n+1} = (P_{n+1}(x_{r,n+1}))^2 = \left\{ (P_n(x))^2 + \frac{(1-x^2)(P'_n(x))^2}{(n+1)^2} \right\}_{x=x_{r,n+1}} = f(x_{r,n+1}) \tag{10}$$

results where

$$f(x) = (P_n(x))^2 + \frac{(1-x^2)(P'_n(x))^2}{(n+1)^2}. \tag{11}$$

But

$$f'(x) = 2P'_n(x)\left[P_n(x) + \frac{(1-x^2)P''_n(x) - xP'_n(x)}{(n+1)^2}\right] = \tag{12}$$

$$= 2P'_n(x)\left[P_n(x) + \frac{xP'_n(x) - n(n+1)P_n(x)}{(n+1)^2}\right] = \frac{2P'_n(x)P'_{n+1}(x)}{(n+1)^2}.$$

Here we have used the differential equation

$$(1-x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0 \tag{13}$$

and the identity [SZEGÖ, (4.7.28), second formula, $\lambda = 1/2$]:

$$xP'_n(x) + (n+1)P_n(x) = P'_{n+1}(x). \tag{14}$$

Consequently $f'(x) < 0$ in the interval between $x_{r,n}$ and $x_{r,n+1}$, i. e. $f(x)$ is decreasing in this interval. Hence

$$\mu^2_{r,n+1} = f(x_{r,n+1}) < f(x_{rn}) = (P_n(x_{rn}))^2 = \mu^2_{r,n}$$

as it was stated above.